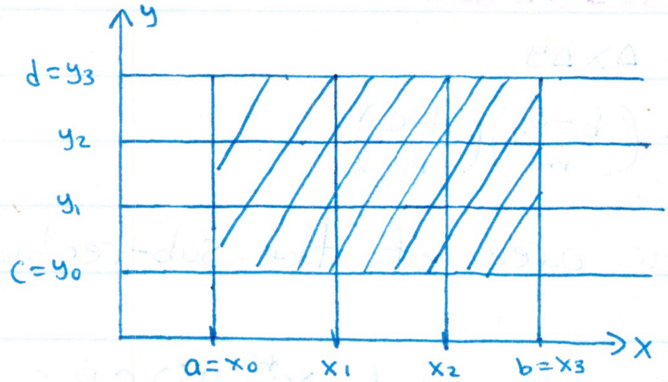
Double Integrals

1. Definitions:



- Let $f: [a,b]\times[c,d] \to \mathbb{R}$
- Choose ints $m, n > 0$
- Divide the interval $[a,b]$ into m equal subintervals $[x_{i-1}, x_i]$. Note that $[a,b]$ has length $b-a$, so each of the subintervals have length $\Delta x = \dfrac{b-a}{m}$.
- Divide the interval $[c,d]$ into n equal subintervals $[y_{j-1}, y_j]$. Note that $[c,d]$ has length $d-c$, so each of the subintervals have length $\Delta y = \dfrac{d-c}{n}$.

- The Rectangle, R, which is equal to $[a,b] \times [c,d]$ becomes $m \times n$ sub-rectangles $[x_{i-1}, x_i] \times [y_{j-1}, y_j] = R_{ij}$.

  I.e. $R = [a,b] \times [c,d]$

- Denote $\Delta A = \Delta x \Delta y$

$$= \left(\frac{b-a}{m}\right)\left(\frac{d-c}{n}\right)$$

  which is the area of the sub-rectangle R_{ij}.

- Choose a sample point $(x_i^*, y_j^*) \in R_{ij}$. Then, $f(x_i^*, y_j^*)\Delta A$ is the vol of the small solid with base $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ and height $f(x_i^*, y_j^*)$.

- $\sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i^*, y_j^*)\Delta A$ approximates the volume of the solid lying under the graph of f and above the rectangle $R = [a,b] \times [c,d]$.

- $$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i^*, y_j^*)\Delta A.$$

  Whenever the limit exists, it's called the Double Integral over R.

- If $f \geq 0$, then $\iint_R f(x,y)\,dA$ is the vol of the solid lying under the graph of f above R.

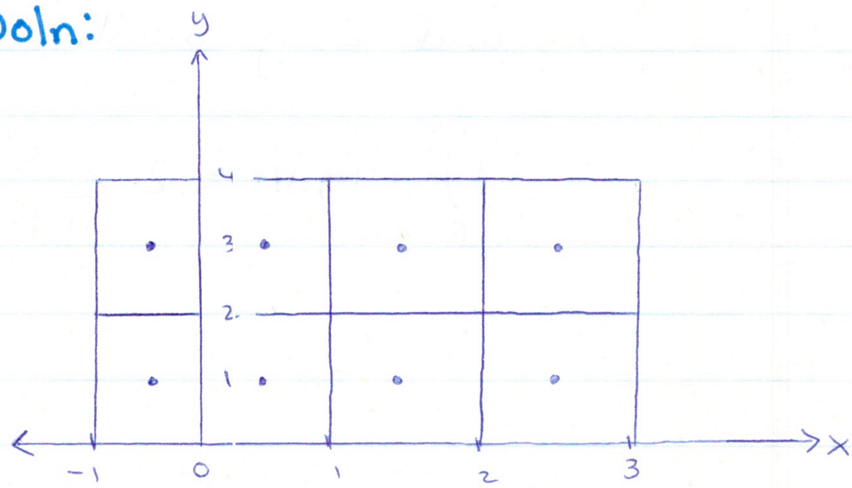
2. Midpoint Rule:

- If we choose the sample points to be the center of the sub-rectangle R_{ij}, that is, $\overline{x_i^*}$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y_j^*}$ is the midpoint of $[y_{j-1}, y_j]$, then we have the midpoint rule:

$$\iint_R f(x,y)\,dA \approx \sum_{j=1}^{n} \sum_{i=1}^{m} f(\overline{x_i^*}, \overline{y_j^*})\,\Delta A$$

- E.g.
  Use the midpoint rule to estimate the volume under $f(x,y) = x^2 + y$ and above the rectangle given by $-1 \le x \le 3$, $0 \le y \le 4$.

Soln:



Each dot represents the midpoint of its rectangle.
The coordinates of the dots are:

1. $(-\frac{1}{2}, 3)$
2. $(\frac{1}{2}, 3)$
3. $(\frac{3}{2}, 3)$
4. $(\frac{5}{2}, 3)$
5. $(-\frac{1}{2}, 1)$
6. $(\frac{1}{2}, 1)$
7. $(\frac{3}{2}, 1)$
8. $(\frac{5}{2}, 1)$

Top Row

Bottom Row

$$\iint_R x^2+y \, dA \approx \sum_{j=1}^{n} \sum_{i=1}^{m} f(\overline{x_i^*}, \overline{y_j^*}) \Delta A$$

ΔA is the area of each small rectangle. $\Delta A = (1)(2) = 2$

Because there are 4 x subdivisions and 2 y subdivisions, $n=2$ and $m=4$.

$$\iint_R x^2+y \, dA \approx 2 \sum_{j=1}^{2} \sum_{i=1}^{4} f(\overline{x_i^*}, \overline{y_j^*})$$

When $j=1$:

$$\sum_{i=1}^{4} f(\overline{x_i^*}, \overline{y_j^*})$$

$$= \sum_{i=1}^{4} f(\overline{x_i^*}, 3) \leftarrow \text{Upper Row}$$

$$= f(-\tfrac{1}{2}, 3) + f(\tfrac{1}{2}, 3) + f(\tfrac{3}{2}, 3) + f(\tfrac{5}{2}, 3)$$

$$= 21$$

When $j=2$:

$$\sum_{i=1}^{4} f(\overline{x_i^*}, \overline{y_j^*})$$

$$= \sum_{i=1}^{4} f(\overline{x_i^*}, 1) \leftarrow \text{Lower Row}$$

$$= f(-\tfrac{1}{2}, 1) + f(\tfrac{1}{2}, 1) + f(\tfrac{3}{2}, 1) + f(\tfrac{5}{2}, 1)$$

$$= 13$$

$$2(21+13) = 68$$

The actual volume is $\frac{208}{3} = 69.\overline{3}$.

$\therefore$ The approximation is fairly close to the actual result.

3. Average Value of a Function:

- The average value of f is denoted by f_{AVE}.

- $f_{AVE} = \dfrac{\iint_R f(x,y)\,dA}{A}$ where A is the area of the rectangle R.

4. Theorems:

- Let $f: R \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a bounded real-valued function over the rectangle R, and suppose that the set of points where f is discont lies on a finite union of graphs of cont functions. Then, f is integrable over R.

- Cont functions are integrable.

5. Properties of Double Integrals:

- Let $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ and $g : R \subset \mathbb{R}^2 \to \mathbb{R}$ be integrable over R.

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i^*, y_j^*)\,\Delta A$$

$$\iint_R g(x,y)\,dA = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} g(x_i^*, y_j^*)\,\Delta A$$

1. Homogeneity:

If c is a constant in $\mathbb{R}$, then

$$\iint_R c f(x,y)\,dA = c \iint_R f(x,y)\,dA$$

2. Linearity:

$$\iint_R (f(x,y) \pm g(x,y))\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

3. Monotonecity:

If $f(x,y) \geq g(x,y)$ on R, then

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$$

4. Additivity:

If R_i, $i=1,2,\dots,m$ are non-overlapping rectangles with

$$\iint_{R_i} f(x,y)\,dA = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i^*, y_j^*)\,\Delta$$

and

$$R = \bigcup_{i=1}^{m} R_i, \text{ then}$$

$$\iint_R f(x,y)\,dA = \sum_{i=1}^{m} \iint_{R_i} f(x,y)\,dA$$

5.

$$\left| \iint_R f(x,y)\,dA \right| \leq \iint_R |f(x,y)|\,dA$$

6. Partial and Iterated Integral:

- Let $f: [a,b] \times [c,d] \to \mathbb{R}$. If we fix y and let x vary from a to b, we can integrate $f(x,y)$ on the interval $[a,b]$ with respect to x. $\int_a^b f(x,y)\,dx$ is called the partial integration with respect to x.

The result is the cross-sectional area that depends on y. This means that $\int_a^b f(x,y)\,dx$ is a function of y, denoted by $A(y)$.

- We may integrate $A(y)$ from c to d to obtain the volume of the solid.

$$V = \int_c^d A(y)\,dy$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

This is called an iterated integral.

- $$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

- Similarly, we can fix x and let y vary from c to d. In this case,

$$V = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

– Fubini's Thm:
Let f be cont on the rectangular region $R = [a,b] \times [c,d]$. Then, the double integral of f over R may be evaluated by either of the 2 iterated integrals.

I.e. $$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$
$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

E.g. Evaluate the integral
$\iint_R xy\,dA$ on $R = [1,2] \times [1,2]$
in 2 ways.

Soln:
1. $$\iint_R xy\,dA = \int_1^2 \int_1^2 xy\,dx\,dy$$
$$= \int_1^2 y \int_1^2 x\,dx\,dy$$
$$= \int_1^2 y \left[\frac{x^2}{2} \Big|_1^2 \right] dy$$
$$= \frac{3}{2} \int_1^2 y\,dy$$
$$= \frac{9}{4}$$

2. $$\iint_R xy\, dA = \int_1^2 \int_1^2 xy \; dy\, dx$$

$$= \int_1^2 x \int_1^2 y \; dy\, dx$$

$$= \int_1^2 x \left[\frac{y^2}{2} \Big|_1^2 \right] dx$$

$$= \frac{3}{2} \int_1^2 x\, dx$$

$$= \frac{9}{4}$$

$$= \int_1^2 \int_1^2 xy \; dx\, dy$$

Note: Sometimes, one of the iterated integrals is easier to solve. Always choose the easiest way to solve, unless otherwise specified.

Note: Let $f(x,y) = g(x)\,h(y)$ and $R = [a,b] \times [c,d]$.

$$\iint_R f(x,y)\, dA = \iint_R g(x)\,h(y)\, dA$$

$$= \int_c^d \int_a^b g(x)\,h(y)\, dx\, dy$$

$$= \left(\int_a^b g(x)\, dx \right) \left(\int_c^d h(y)\, dy \right)$$

E.g.

We know that $\int_1^2\int_1^2 xy\,dx\,dy = \frac{9}{4}$.

$$\left(\int_1^2 x\,dx\right)\left(\int_1^2 y\,dy\right)$$

$$= \left(\tfrac{1}{2}\right)\left(x^2\Big|_1^2\right)\left(\tfrac{1}{2}\right)\left(y^2\Big|_1^2\right)$$

$$= \frac{9}{4}$$

7. Double Integrals Over General Regions:

- Let $f: D \subset \mathbb{R}^2 \to \mathbb{R}$ be a cont function and choose a rectangle R that contains the region D.

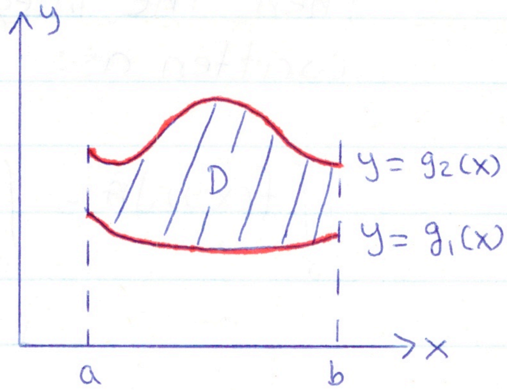
$$f^*(x,y) = \begin{cases} f(x,y) & ,\text{if } (x,y) \in D \\ 0 & ,\text{if } (x,y) \notin D \text{ and } (x,y) \in R \end{cases}$$

- The integral of f over the region D is given by:

$$\iint_D f(x,y)\,dA = \iint_R f^*(x,y)\,dA$$

- There are 2 types of regions we need to look at.

1\.

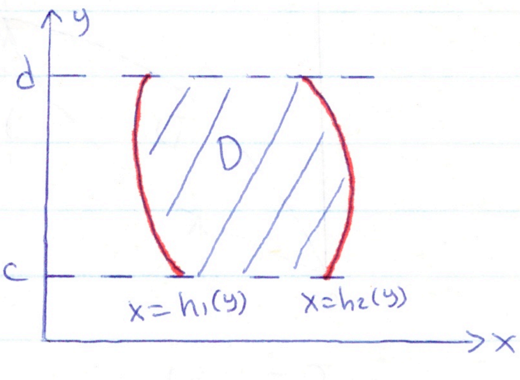


This is called y-simple.

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

2\.



This is called x-simple.

$$D = \{(x,y) \mid h_1(y) \le x \le h_2(y),\ c \le y \le d\}$$

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

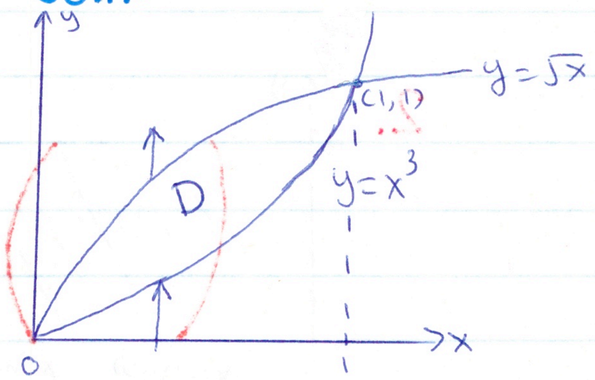
Note: If D can be split into separate regions $D_1, D_2, \ldots D_n$, then the integral can be written as:

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA + \cdots \iint_{D_n} f(x,y)\,dA$$

- E.g. Evaluate each of the following integrals.

1. $\iint_D 4xy - y^3\,dA$, D is the region bounded by $y=\sqrt{x}$ and $y=x^3$.

Soln:



$0 \le x \le 1$

$x^3 \le y \le \sqrt{x}$

$$\iint_D 4xy - y^3\,dA = \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy - y^3\,dy\,dx$$

$$= \int_0^1 4x\left[\frac{y^2}{2}\right] - \frac{y^4}{4}\Big|_{x^3}^{\sqrt{x}}\, dx$$

$$= \int_0^1 2xy^2 - \frac{y^4}{4}\Big|_{x^3}^{\sqrt{x}}\, dx$$

$$= \int_0^1 \frac{7}{4}x^2 - 2x^7 + \frac{x^{12}}{4}\, dx$$

$$= \left(\frac{7}{12}x^3 - \frac{1}{4}x^8 + \frac{1}{52}x^{13}\Big|_0^1\right)$$

$$= \frac{55}{156}$$

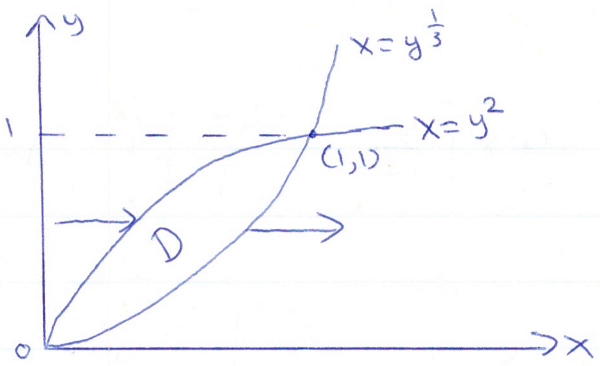
Here, we used y-simple. However, we can calculate the integral using x-simple, too.

We know that the region is bounded by

1. $y = \sqrt{x}$
2. $y = x^3$

However, we can rewrite it in terms of x.

1. $x = y^2$
2. $x = \sqrt[3]{y}$



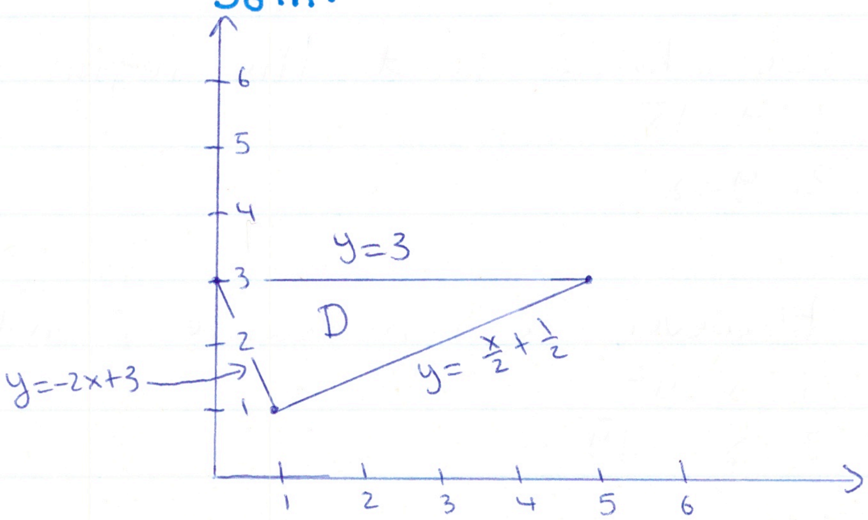
$$\iint_D 4xy - y^3 \, dA = \int_0^1 \int_{y^2}^{\sqrt[3]{y}} 4xy - y^3 \, dx\, dy$$

$$= \frac{55}{156}$$

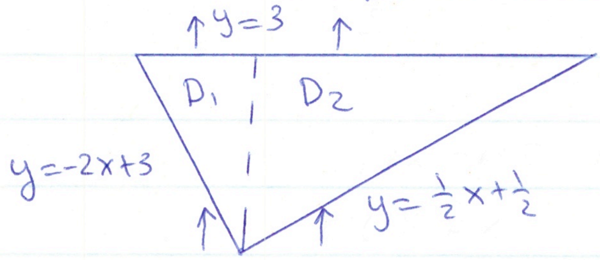
We got the same answer as before, but used x-simple.

2. $\iint_D 6x^2 - 40y \, dA$ where D is the triangle with vertices (0,3), (1,1) and (5,3).

Soln:



We have to split the triangle into 2 parts.



$$D_1 = \{(x,y) \mid 0 \le x \le 1,\ -2x+3 \le y \le 3\}$$

$$D_2 = \{(x,y) \mid 1 \le x \le 5,\ \tfrac{x}{2}+\tfrac{1}{2} \le y \le 3\}$$

$$\iint_D 6x^2-40y\,dA = \iint_{D_1} 6x^2-40y\,dA + \iint_{D_2} 6x^2-40y\,dA$$

$$= \int_0^1 \int_{-2x+3}^{3} 6x^2-40y\,dy\,dx + \int_1^5 \int_{\frac{x}{2}+\frac{1}{2}}^{3} 6x^2-40y\,dy\,dx$$

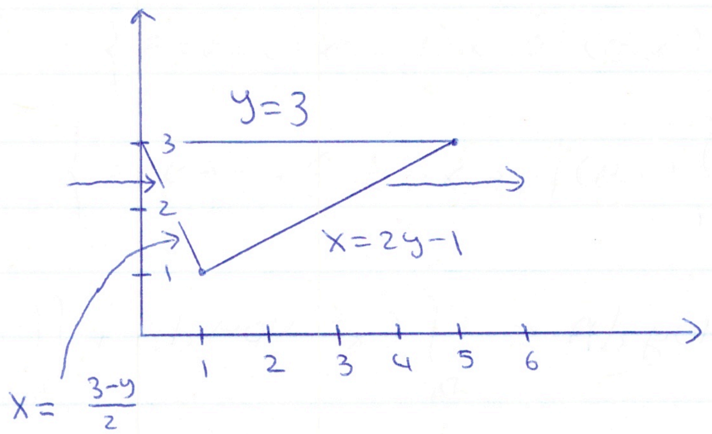
$$= -\frac{935}{3}$$

This is y-simple.

If we calculated the integral with x-simple, we don't have to split the triangle. This is because the same function is always on the left and the same function is always on the right.

$$y = -2x + 3 \rightarrow y - 3 = -2x$$
$$x = \frac{3-y}{2}$$

$$y = \frac{x}{2} + \frac{1}{2} \rightarrow 2y = x + 1$$
$$x = 2y - 1$$



$$\iint_D 6x^2 - 40y \, dA = \int_1^3 \int_{\frac{3-y}{2}}^{2y-1} 6x^2 - 40y \, dx\,dy$$

$$= -\frac{935}{3}$$

– Thm: Double integrals over regions can be represented as iterated integrals in 2 ways if D is both x-simple and y-simple. Sometimes, one way is easier to solve than the other.

8. Geometric Interpretations:

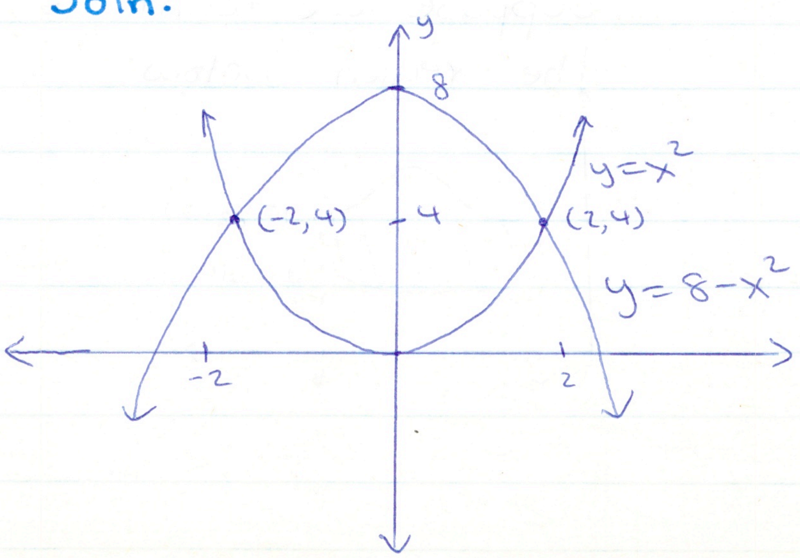
There are 2 geometric interpretations of the double integral.

1. The volume of the solid that lies below the surface given by $z = f(x,y)$ and above the region D in the xy-plane is given by:

$$V = \iint_D f(x,y)\, dA$$

E.g. Find the vol of the solid that lies beneath the surface given by $z = 16xy + 200$ and lies above the region in the xy-plane bounded by: $y = x^2$ and $y = 8 - x^2$.

Soln:



$-2 \le x \le 2$

$x^2 \le y \le 8 - x^2$

$$V = \iint_D 16xy + 200 \, dA$$

$$= \int_{-2}^{2} \int_{x^2}^{8-x^2} 16xy + 200 \quad dy\,dx$$

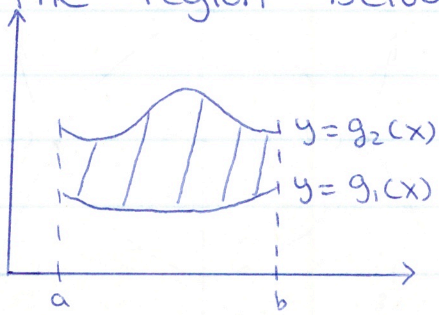
$$= \frac{12800}{3}$$

Note: Regions below the xy-plane have negative volume and regions above the xy-plane have positive area.

2. Area of $D = \iint_D 1 \, dA = \iint_D dA$

Explanation:

Suppose we want to find the area of the region below:



From A36/A37, we know that the area can be found by the integral

$$A = \int_a^b g_2(x) - g_1(x)\, dx$$

In terms of double integrals:

$$A = \iint_D dA$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} 1 \, dy\, dx$$

$$= \int_a^b y \Big|_{g_1(x)}^{g_2(x)} dx$$

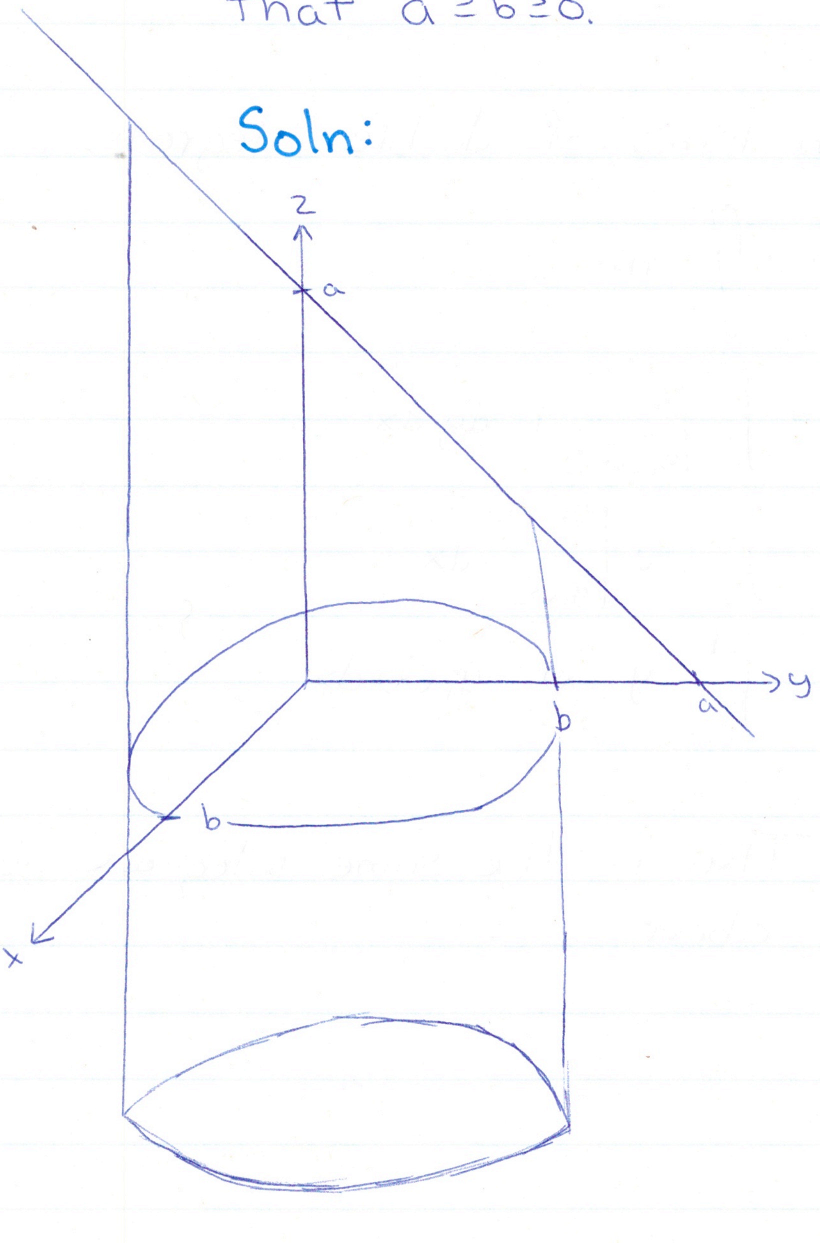
$$= \int_a^b g_2(x) - g_1(x)\, dx$$

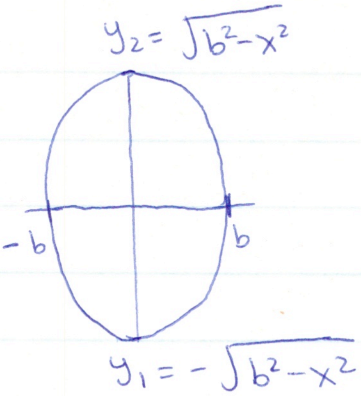
This is the same integral as above.

9. Examples:

1. Calculate the vol within the cylinder $x^2 + y^2 = b^2$ between the plane $y+z=a$ and $z=0$ given that $a \geq b \geq 0$.

Soln:





$$V = \int_{-b}^{b} \int_{-\sqrt{b^2-x}}^{\sqrt{b^2-x}} (a-y)\, dy\, dx$$

$$= a\pi b^2$$

2. Find the volume enclosed by the planes $4x + 2y + z = 10$, $y = 3x$, $z = 0$, and $x = 0$.

Soln:

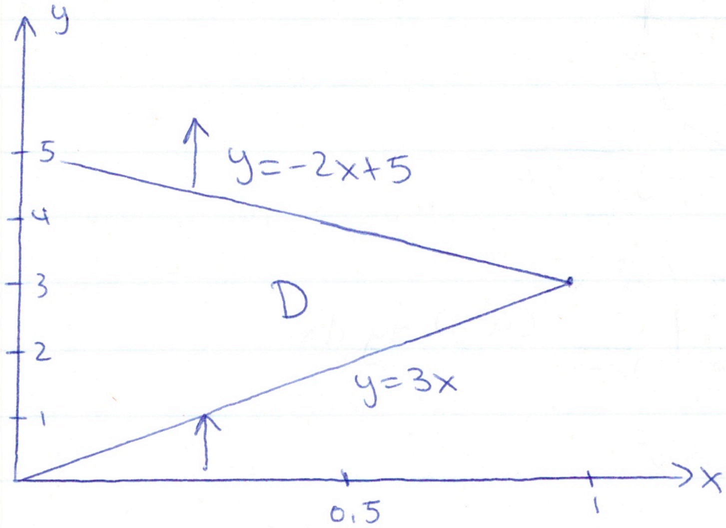
$z = 10 - 4x - 2y$

We are looking for the vol under $z = 10 - 4x - 2y$ and above the region D in the xy-plane.

The region D will be the region in the xy-plane (I.e. $z=0$) that is bounded by $y = 3x$, $x = 0$ and the line where $z = 10 - 4x - 2y$ intersects the xy-plane. To find this line, let $z = 0$ and solve.

$0 = 10 - 4x - 2y$

$y = -2x + 5$



$0 \le x \le 1$

$3x \le y \le -2x + 5$

$$V = \int_0^1 \int_{3x}^{-2x+5} 10 - 4x - 2y \; dy \, dx$$

$$= \frac{25}{3}$$

Note: $V = \iint_D f(x,y)\,dA$ gives the net volume between the graph $z = f(x,y)$ and the region D in the xy-plane.